Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(00)00102-7
J. Appl. Maths Mechs, Vol. 64, No. 5, pp. 729–738, 2000

0021-8928/00/$—see front matter

THE GYROSCOPIC STABILITY OF THE TRIANGULAR STATIONARY SOLUTIONS OF THE GENERALIZED PLANAR THREE-BODY PROBLEM†

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(*Received* 8 *July* 1999)

The gyroscopic stability of the triangular solutions is investigated for the generalized planar three-body problem (point masses) which differs from the classical three-body problem by the addition of a weightless elastic tether connecting two of the three bodies. It is shown that, when the tether has low stiffness and the masses of the connected bodies are substantially different, the triangular motions are stable for any values of the remaining parameters of the system.

The gyroscopic stability of the motions under consideration in the case of a rigid connection and equal masses of the connected bodies has been investigated previously [1]. In particular, a diagram showing the distribution of the fields of gyroscopic stability was presented which shows that, in the satellite domain of system parameters, triangular solutions are unstable and are therefore of limited interest in space technology.

In the proposed investigation, the rigid connection is replaced by an elastic tether and the change in this diagram is traced as a function of the coefficient of elasticity of the tether and the ratio of the magnitudes of the terminal masses. It is found that, as the stiffness of the tether is reduced, the gyroscopic stability domain increases, it is shifted towards the range of parameters characteristic of satellites and covers this domain when the value of the coefficient of elasticity is below a certain critical value. Hence, in the case when the connection is sufficiently elastic, triangular motions can be of interest in the design of two-mass tethered satellite systems. An inequality in the masses of the connected bodies is also found to have a positive effect on the stability. In the case of very unequal masses and a sufficiently low stiffness of the tether, the triangular motions become gyroscopically stable for any values of the remaining parameters.

Neglecting the mutual Newtonian attraction of the connected bodies, the calculated tension in the tether is found to be positive. At a first glance, it follows from this that the connection can be realized using a tether. However, when the Newtonian attraction between these bodies is additionally taken into account, the tension is found to be positive only in the case of a fairly long tether. The tension is negative in the case of shorter tethers and, for the physical realization of the connection in this case, it is necessary to use small thrusters (compressed gas cylinders, for example) at the ends of the tether which create two equal, constant stretching forces acting along the tether.

The stability and bifurcation of the steady motions in the planar problem of two point masses connected by a weightless spring in a central Newtonian gravitational field have been studied in [2]. The steady spatial motions of such a system were considered in [3] and the sufficient conditions for the stability of one of the rectilinear configurations were obtained. The sufficient and necessary conditions for the stability of the steady motions in the formulation adopted in this paper were obtained in [4].

The problem of constructing controls, which ensure the coupling of the elements of extended space stations using the creation of artificial forces which simulate elastic and viscoelastic forces, has been discussed in several papers (see [5], for example, and the references therein).

†*Prikl. Mat. Mekh.* Vol. 64, No. 5, pp. 761–771, 2000.

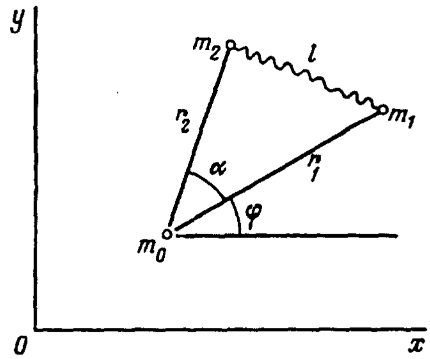

Fig. 1

1. FORMULATION OF THE PROBLEM AND THE EQUATIONS OF MOTION

Suppose that three point masses m_0, m_1 and m_2 move in a fixed plane which is defined by the system of coordinates Oxy (Fig. 1), (x, y) are the coordinates of the point m_0, r_1 and r_2 are the distances from the point m_0 to the points m_1 and m_2, φ is the angle between the Ox axis and the vector m_0m_1 and α is the angle between m_0m_1 and m_0m_2. The coordinates of the points m_1 and m_2 are defined by the formulae $(x + r_1c_1, y + r_1s_1)$ and $(x + r_2c_2, y + r_2s_2)$, where

$$c_1 = \cos\varphi, \quad c_2 = \cos(\varphi+\alpha), \quad s_1 = \sin\varphi, \qquad s_2 = \sin(\varphi+\alpha), \quad c = \cos\alpha, \quad s = \sin\alpha$$

For the generalized velocities, we introduced the formulae

$$u_1 = \dot{x}, \quad u_2 = \dot{y}, \quad u_3 = \dot{\varphi}, \quad v_1 = \dot{r}_1, \quad v_2 = \dot{r}_2, \quad v_3 = \dot{\alpha}$$

Twice the kinetic energy of the system, the potential energy of the interaction of the points m_1 and m_2 and the potential energy of the interaction of the point m_0 with m_1 and m_2 have the form

$$\begin{aligned}2T &= m_0(u_1^2 + u_2^2) + m_1[(v_1 + u_1c_1 + u_2s_1)^2 + (u_3r_1 - u_1s_1 + u_2c_1)^2] + \\ &+ m_2[(v_2 + u_1c_2 + u_2s_2)^2 + ((u_3 + v_3)r_2 - u_1s_2 + u_2c_2)^2] \\ E &= E(l), \quad \Pi = -\gamma m_0(m_1/r_1 + m_2/r_2); \quad l = \sqrt{r_1^2 + r_2^2 - 2r_1r_2c}\end{aligned} \tag{1.1}$$

Here, γ is the Newtonian gravitational constant and l is the distance between the points m_1 and m_2.

Henceforth, we shall not specificially define the function $E(l)$. It is sufficient to know the values of its first and second derivatives (E' and E'') with respect to l. Hence, all the results can be interpreted in terms of an arbitrary force interaction between the bodies m_1 and m_2. In particular, if $E = -\gamma m_1m_2/l$, we shall have the classical problem of three mutually gravitating bodies. In this case, the quantity E'' is negative. It therefore makes sense to consider both positive and negative values of E''. However, we will restrict ourselves to the case when $E'' \geqslant 0$ in this investigation.

The kinetic and potential energies do not depend explicitly on x and y, and the equations of motion of the system therefore admit of cyclic integrals which express the conservation of momentum of the system

$$\begin{aligned}\partial T/\partial u_1 &= mu_1 - (m_1r_1s_1 + m_2r_2s_2)u_3 + m_1c_1v_1 + m_2c_2v_2 - m_2r_2s_2v_3 = 0 \\ \partial T/\partial u_2 &= mu_2 + (m_1r_1c_1 + m_2r_2c_2)u_3 + m_1s_1v_1 + m_2s_2v_2 + m_2r_2s_2v_3 = 0\end{aligned} \tag{1.2}$$

where $m = m_0 + m_1 + m_2$ is the mass of the whole system. Without loss of generality, we shall put the constants in the integrals equal to zero, assuming that the system of coordinates Oxy moves translationally with the centre of mass of the system.

Following Routh's method, we will eliminate the cyclic velocities u_1 and u_2. In order to do this, we solve linear system (1.2) for u_1 and u_2 and substitute this solution into T. We obtain

$$
\begin{aligned}
&2T^* = Ju_3^2 + 2g_4u_3(r_2\upsilon_1 - r_1\upsilon_2) + 2(g_2r_2 - g_3r_1)r_2u_3\upsilon_3 + g_1\upsilon_1^2 + g_2(\upsilon_2^2 + r_2^2\upsilon_3^2) - \\
&-2g_3\upsilon_1\upsilon_2 + 2g_4r_2\upsilon_1\upsilon_3 \\
&J = g_1r_1^2 + g_2r_2^2 - 2g_3r_1r_2 \\
&g_1 = \frac{m_1(m_0 + m_2)}{m}\ (1 \leftrightarrow 2), \quad g_3 = \frac{m_1m_2}{m}c, \quad g_4 = \frac{m_1m_2}{m}s
\end{aligned} \tag{1.3}
$$

Henceforth, the symbol $(1 \leftrightarrow 2)$ denotes the existence of a further relation which differs by a cyclic permutation of the indices 1 and 2.

The quantity T^* corresponds to the kinetic energy of the reduced system, which is equal to the kinetic energy of the relative motion of the point masses m_0, m_1, m_2, in the König system of coordinates Gxy with its origin at the centre of mass G of the system, and J is the moment of inertia of the system with respect to the centre of mass G.

In this system, u_3 becomes the cyclic velocity. The cyclic integral, which expresses the conservation of angular momentum of the points of the system with respect to the centre of mass G

$$
\partial T^* / \partial u_3 = Ju_3 + g_4(r_2\upsilon_1 - r_1\upsilon_2) + (g_2r_2 - g_3r_1)r_2\upsilon_3 = p \tag{1.4}
$$

corresponds to it, whence

$$
Ju_3 = p - g_4(r_2\upsilon_1 - r_1\upsilon_2) - (g_2r_2 - g_3r_1)r_2\upsilon_3 \tag{1.5}
$$

Let us ignore the cyclic velocity u_3 using Routh's method. In order to do this, we introduce the function

$$
\begin{aligned}
&T^{**} = T^* - \frac{\partial T^*}{\partial u_3}u_3 = T^*\big|_{u_3=0} - T^*\big|_{\upsilon_1=\upsilon_2=\upsilon_3=0} = \frac{g_1}{2}\upsilon_1^2 + \frac{g_2}{2}(\upsilon_2^2 + r_2^2\upsilon_3^2) \\
&-g_3\upsilon_1\upsilon_2 + g_4r_2\upsilon_1\upsilon_3 - \frac{J}{2}u_3^2
\end{aligned} \tag{1.6}
$$

In the last term of formula (1.6), instead of the cyclic velocity u_3, its value from (1.5) has to be substituted.

In constructing Routh's function

$$
R = T^{**} - \Pi - E \tag{1.7}
$$

we write the equations of motion in the form

$$
\frac{d}{dt}\frac{\partial R}{\partial \upsilon_i} - \frac{\partial R}{\partial q_i} = 0, \quad i = 1, 2, 3, \quad q = (r_1, r_2, \alpha) \tag{1.8}
$$

The function R can be represented in the form of the sum $R = R_2 + R_1 + R_0$, where R_i are forms of degree of homogeneity i which are homogeneous with respect to the positional velocities υ_1, υ_2, υ_3. In terms of the reduced system, it can be said that the term R_2 represents the kinetic energy and that $(-R_0)$ represents the amended potential energy W, that is,

$$
W = -R_0 = \frac{p^2}{2J} + \Pi + E(l) \tag{1.9}
$$

and the linear term R_1 determines the gyroscopic forces.

The equations of motion (1.8) admit of a Painlevé–Jacobi integral $R_2 + W = h$.

2. STEADY MOTIONS

On the basis of Routh's theorem, the steady motions are defined as the critical points of the amended potential energy W, that is, as the solutions of the system of equations

$$\frac{\partial W}{\partial r_1} = -\frac{p^2}{J^2}\left[\frac{m_0 m_1}{m} r_1 + \frac{m_1 m_2}{m}(r_1 - r_2 c)\right] +$$

$$+\gamma \frac{m_0 m_1}{r_1^2} + \frac{r_1 - r_2 c}{l} E' = 0 \quad (1 \leftrightarrow 2) \tag{2.1}$$

$$\frac{\partial W}{\partial \alpha} = -\frac{p^2}{J^2}\frac{m_1 m_2}{m} r_1 r_2 s + \frac{r_1 r_2}{l} E' s = 0$$

These equations possess triangular solutions for which the points m_1 and m_2 are at the same distance from the point m_0 and

$$E' = \frac{m_1 m_2}{m}\frac{p^2 l}{J^2}, \quad r_1 = r_2 = r = \frac{p^2}{m\gamma(g_1 + g_2 - 2g_3)^2} \tag{2.2}$$

If $E' = -\gamma m_1 m_2 / l^3$, that is, the points m_1 and m_2 are attracted in accordance with the law of universal gravitation, then $r = l$ and the solution is identical to Lagrange's classical triangular solution.

3. THE NECESSARY CONDITIONS FOR STABILITY

We write the equations in variations in the form

$$\frac{d}{dt}\frac{\partial \tilde{R}}{\partial v_i} - \frac{\partial \tilde{R}}{\partial \xi_i} = 0; \quad \tilde{R} = \frac{1}{2} v^T A v + v^T D \xi - \frac{1}{2}\xi^T C \xi \tag{3.1}$$

where $\tilde{R}$ is the quadratic part of Routh's function R in the relative velocities $\upsilon_1, \upsilon_2, \upsilon_3$ and the deviations $\xi_i = q_i - q_i^0$ of the variables q_i from their values q_i^0 in the unperturbed steady motion and C is the matrix of the part $\tilde{W}$ of the amended potential energy W which is quadratic in the deviations ξ_1, ξ_2, ξ_3, that is

$$\tilde{W} = \frac{1}{2}\xi^T C \xi = \frac{1}{2}\delta^2 W = \frac{1}{2}\left(p^2 \delta^2 \frac{1}{2J} + \delta^2 \Pi + \delta^2 E\right) \tag{3.2}$$

In order to calculate the elements c_{ij} ($c_{ij} = c_{ji}$) of matrix C, we write out auxiliary formulae for the variations of certain functions occurring in relation (3.2). We introduce the notation

$$J_1 = g_1 r_1 - g_3 r_2 \ (1 \leftrightarrow 2), \quad J_3 = g_4 r_1 r_2$$

$$l_1 = r_1 - r_2 c \ (1 \leftrightarrow 2), \quad l_3 = r_1 r_2 s$$

Then (here and everywhere hence summation is carried out over repeated indices from 1 to 3)

$$\delta J = 2J_i \xi_i, \quad \delta^2 J = 2\xi_i \delta J_i; \quad l\delta l = l_i \xi_i, \quad l^3 \delta^2 l = l^2 \xi_i \delta l_i - (l\delta l)^2$$

$$\delta J_1 = g_1 \xi_1 - g_3 \xi_2 + g_4 r_2 \xi_3 \ (1 \leftrightarrow 2), \quad \delta J_3 = g_4 r_2 \xi_1 + g_4 r_1 \xi_2 + g_3 r_1 r_2 \xi_3$$

$$\delta l_1 = \xi_1 - c\xi_2 + r_2 s \xi_3 \ (1 \leftrightarrow 2), \quad \delta l_3 = r_2 s \xi_1 + r_1 s \xi_2 + r_1 r_2 c \xi_3 \tag{3.3}$$

$$J^3 \delta^2 \frac{1}{2J} = (\delta J)^2 - \frac{J}{2}\delta^2 J, \quad \delta^2 E = \left(\frac{E''}{l^2} - \frac{E'}{l^3}\right)(l\delta l)^2 + \frac{E'}{l}\xi_i \delta l_i$$

On substituting the expressions which have been obtained for δJ, $\delta^2 J$, $l\delta l$, and δl_1, δl_2, δl_3, into the last formula of (3.3) and, then, into (3.2), we find that

$$c_{11} = p'(4J_1^2 - Jg_1) + e' + e'' l_1^2 - 2\gamma m_0 m_1 / r_1^3 \ (1 \leftrightarrow 2)$$

$$c_{33} = p'(4J_3^2 - Jg_3 r_1 r_2) + e' r_1 r_2 c + e'' l_3^2$$

$$c_{12} = p'(4J_1 J_2 + Jg_3) - e'c + e'' l_1 l_2 \tag{3.4}$$

$$c_{13} = p'(4J_1J_3 - Jg_4r_2) + e'r_2s + e''l_1l_3 \quad (1 \leftrightarrow 2)$$
$$p' = p^2 / J^3, \quad e' = E'/l, \quad e'' = (E'' - e')/l^2$$

The matrices A and D are determined from the expressions for R_2 and R_1 which are obtained from formulae (1.7) and (1.6)

$$A = \frac{1}{J}\begin{Vmatrix} Jg_1 - g_4^2r_2^2 & -Jg_3 + g_4^2r_1r_2 & J_1J_3 \\ -Jg_3 + g_4^2r_1r_2 & Jg_2 - g_4^2r_1^2 & J_2J_3 \\ J_1J_3 & J_2J_3 & Jg_2r_2^2 - J_2^2r_2^2 \end{Vmatrix}$$
$$D = \frac{p}{J^2}\begin{Vmatrix} -2J_1g_4r_2 & -2J_2g_4r_2 + Jg_4 & -2J_3g_4r_2 + Jg_3r_2 \\ 2J_1g_4r_1 - Jg_4 & 2J_2g_4r_1 & 2J_3g_4r_1 - Jg_3r_1 \\ -2J_1J_2r_2 - Jg_3r_2 & -2J_2^2r_2 + J(J_2 + g_2r_2) & -2J_2J_3r_2 + Jg_4r_1r_2 \end{Vmatrix} \tag{3.5}$$

In matrix notation, Eqs (3.1) take the form

$$A\ddot{\xi} + B\dot{\xi} + C\xi = 0$$
$$B = D - D^T = 2\frac{m_0m_1m_2p}{mJ^2}r_1r_2\begin{Vmatrix} 0 & 0 & r_2 \\ 0 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{Vmatrix}$$

The characteristic equation reduces to the form

$$\det(A\lambda^2 + B\lambda + C) = a_0z^3 + a_1z^2 + a_2z + a_3 = 0; \quad z = \lambda^2$$
$$a_0 = \det A, \quad a_3 = \det C, \quad a_1 = c_{ij}A_{ij} + a_{ij}B_{ij}, \quad a_2 = a_{ij}C_{ij} + c_{ij}B_{ij} \tag{3.6}$$

where A_{ij}, B_{ij}, C_{ij} are the cofactors of the elements a_{ij}, b_{ij}, c_{ij} of the matrices A, B and C.

For stability, it is necessary that all the roots of Eq (3.6) with respect to the variable z should be real and negative. A check was carried out using Sturm series for the interval $(-\infty, 0)$. If the number of real roots in this interval is equal to three, the steady motion is stable in the linear approximation. This condition leads to the inequalities

$$a_2 > 0, \quad a_3 > 0, \quad a_4 = a_1^2 - 3a_0a_2 > 0$$
$$a_5 = a_1a_2 - 9a_0a_3 > 0, \quad a_6 = a_5(4a_1a_4 - 3a_0a_5) - 4a_2a_4^2 > 0 \tag{3.7}$$

If the left-hand side of just one of inequalities (3.7) is negative, the steady motion is unstable. It is not difficult to show that, among conditions (3.7), the inequalities $a_4 > 0$, $a_5 > 0$ are not independent and can be discarded.

4. ANALYSIS OF THE NECESSARY CONDITIONS FOR STABILITY

The necessary conditions for stability (3.7) of triangular steady motions, which are defined by formulae (2.2), depend on the nine parameters r, α, m_0, m_1, m_2, γ, p, E', E'' The number of parameters can be reduced bu changing to dimensionless quantities by means of a special choice of the units of measurement, which depend on part of these parameters. For simplicity, we shall retain the old notation of the dimensional quantities for the dimensionless quantities but denote the dimensional quantities by an upper bar. We select the unit of mass as $\overline{M}$, the unit of time as $\overline{T}$ and the unit of length as $\overline{L}$:

$$\overline{M} = \overline{m}_1 + \overline{m}_2, \quad \overline{T} = 1/\overline{\omega} = \overline{J}/\overline{p}, \quad \overline{L} = \overline{r} = \overline{T}^2/(\overline{m}\overline{\gamma})$$

where $\overline{\omega}$ is the angular velocity of the steady rotation of the whole system as a rigid whole in the steady motion being considered. With this choice of units of measurement, the dimensionless parameters m_2, p, γ, can be eliminated using the relations

$$m_2 = 1 - m_1, \quad p = J, \quad \gamma = 1/m$$

The parameter E' is eliminated using the first of the equations for the steady motions (2.2), that is, in the new variables

$$E' = m_1 m_2 l/m$$

Furthermore, from formulae (2.2) and the last three formulae of (3.4), we obtain

$$p' = 1/J, \quad e' = m_1 m_2/m, \quad e'' = (E'' - e')/l^2 = (k-1)e'/l^2$$

where $k = E''/e'$ is the magnitude of E'' scaled by the quantity e'.

After this has been done, the four dimensionless parameters α, m_0, m_1 and k remain in stability conditions (3.7). As a result, all of the formulae are simplified. For example, the elements of the matrix C (3.4) take the form

$$c_{11} = 4J_1^2/J + e''l_1^2 - 3m_0 m_1/m \quad (1 \leftrightarrow 2), \quad c_{33} = 4J_3^2/J + e''l_3^2$$
$$c_{12} = 4J_1 J_2/J + e''l_1 l_2, \quad c_{13} = 4J_1 J_3/J + e''l_1 l_3 \ (1 \leftrightarrow 2)$$

Here

$$J = g_1 + g_2 - 2g_3, \quad J_1 = g_1 - g_3, \quad J_2 = g_2 - g_3, \quad J_3 = g_4$$
$$e'' = m_1 m_2 (k-1)/(2ml_1), \quad m_2 = 1 - m_1, \quad m = m_0 + 1$$
$$l_1 = l_2 = 1 - \cos\alpha, \quad l_3 = \sin\alpha$$

g_1, g_2, g_3, g_4, are defined by the last three formulae of (1.3).

Introducing the notation $q = m_1, m_2$, after dividing by the general factor

$$[m_0 q(m_0 + 2ql_1)]^2$$

we reduce the coefficients of characteristic equation (3.6) to the form

$$a_0 = (m_0 + 1)^2, \quad a_1 = (m_0 + 1)(2m_0 + k + 4)$$
$$a_2 = 2m_0^2 + m_0(3kl_1 - 4k - 3l_1 + 14) + 2(k - 9ql_1^2 + 18ql_1 + 3)$$
$$a_3 = 3(2 - l_1)[-m_0(k-1) + 6ql_1(k+3)]$$

As analytic, numerical and graphical investigations carried out when $k \geqslant 0$ show, the stability conditions (3.7) reduce to the two inequalities

$$a_3 > 0, \quad a_6 > 0 \tag{4.1}$$

where a_3 is the discriminant of the quadratic part of the amended potential energy $\tilde{W}$ and a_6 is the discriminant of Eq. (3.6).

For a graphical representation of the results, we chose the parameters α, m_0, m_1 and k as the basis where

$$0 \leqslant \alpha \leqslant \pi, \quad 0 \leqslant m_0 \leqslant \infty, \quad 0 \leqslant m_1 \leqslant 1/2, \quad 0 \leqslant k \leqslant \infty \tag{4.2}$$

but, in order to write the formulae in a more compact form, instead of α and m_1, we use the parameters $l_1 = 1 - \cos\alpha$ and q, which vary within the ranges

$$0 \leqslant l_1 \leqslant 2, \quad 0 \leqslant q \leqslant 1/4$$

The first condition from (4.1) is equivalent to the inequality

$$(k-1)m_0 \leqslant 6(k+3)ql_1$$

In particular, when $k \leqslant 1$, this condition is satisfied for all values of the remaining parameters from the domain of definition (4.2).

The left-hand side of the second condition has a more complex from. It is a sixth-degree polynomial in l_1, a fourth-degree polynomial in m_0 and a fourth-degree polynomial in k. While this polynomial is not written out here in its expanded form, we shall subsequently indicate some of its properties, which determine the form of the stability domain.

5. THE LIMITING CASE

We will consider the limiting case $k = \infty$, $m_1 = 1/2$ $(q = 1/4)$, when the point masses m_1, m_2 form a rigid dumbbell with different masses at the ends [1].

In order to map the infinite range of possible values of the parameters m_0 $(0 < m_0 < \infty)$ into the finite interval [0,2], instead of m_0, we introduce the new parameter [1]

$$\tilde{m}_0 = \begin{cases} m_0^{-1}, & m_0 > 1 \\ 2 - m_0, & m_0 \leqslant 1 \end{cases}$$

In this manner, we shall construct the stability domain in the rectangle $\{\alpha, \tilde{m}_0) \in [0, \pi] \times [0, 2]\}$.

In this limiting case, the stability conditions, obtained from the coefficients of the highest power of k on the left-hand sides of inequalities (4.1), take the form

$$m_0 < 3l_1/2, \quad m_0^2(9l_1^2 - 48l_1 + 64) + 4m_0(9l_1^2 - 21l_1 + 8) + 4(9l_1^2 - 18l_1 + 1) > 0 \tag{5.1}$$

Substituting $m_0 = 3l_1/2$ into the left-hand side of the second inequality, we obtain

$$(81/4)(l_1 - 2/3)^4$$

Hence it follows that, in this limiting case, the curve $a_3 = 0$ touches the curve $a_6 = 0$ at the point

$$l_1 = 2/3(\cos\alpha = 1/3), \quad m_0 = 1$$

and, with the exception of this point, is wholly located in the domain $a_6 > 0$. The region of stability, which is bounded by the curves $a_3 = 0$ and $a_6 = 0$, is shown hatched in the upper part of Fig. 2.

In the previously adopted notation [1]

$$\beta^2 = \frac{1 - \cos\alpha}{1 + \cos\alpha}, \quad \mu_0 = 1 + \frac{1}{m_0}$$

the stability conditions (5.1) take the form

$$\begin{gathered} \mu_0 - 1 > \frac{1}{3}\left(1 + \frac{1}{\beta^2}\right) \\ (\beta^4 - 34\beta^2 + 1)(\mu_0 - 1)^2 + 2(\beta^4 - 13\beta^2 + 4)(\mu_0 - 1) + (\beta^4 + 8\beta^2 + 16) > 0 \end{gathered} \tag{5.2}$$

The first inequality of (5.2) is identical to the first of the conditions in [1]. There is an inaccuracy in the second condition in [1] (the 15 in the last bracket should be 16). On account of this, the curves in [1], defining the stability domain, intersect at two points instead of touching, which is incorrect. It should be noted here that, prior to the publication of [1], there was a preliminary publication by the same authors† in which an analytic expression for the second of the stability conditions had still not been obtained, but the touching of the curves is clearly observable on the numerically constructed graph. The second inequality can be rewritten in expanded form with respect to $m_0 = 1/\mu_0 - 1)$ as

$$m_0 > \frac{-(\beta^4 - 13\beta^2 + 4) + 12\sqrt{3}\beta\sqrt{\beta^2 + 1}}{\beta^4 + 8\beta^2 + 16}$$

†BELETSKII, V. V. and PONOMAREVA, O. N., Parametric analysis of the stability of relative equilibrium in a gravitational field. Preprint No. 12, The M. V. Keldysh Institute of Applied Mathematics, Academy of Sciences of the USSR, 1988.

6. THE DEPENDENCE OF THE STABILITY DOMAIN ON THE PARAMETERS

We will now study the change in the stability domain, constructed for the case when $k = \infty, m_1 = 1/2$, when the parameters k and m_1 are reduced from their limiting values of ∞ and 1/2. We shall map the stability domains onto the same plane $(\alpha, \tilde{m}_0)$ for various fixed values of the parameters k and m_1. We recall that k is the ratio of the coefficient of elasticity of the tether to the magnitude of e' and the coefficient of elasticity E'', for a fixed k, therefore depends on the parameters m_1, m_0, α. The formula for calculating the parameter E'' is given below in Section 7.

Substituting the value of m_0 from the equation $a_3 = 0$, that is

$$m_0 = 6(k+3)(k-1)^{-1}ql_1$$

into a_6, in the general case, we obtain the expression

$$\frac{(k-1)^8}{9(m_0+1)^4}a_6\Big|_{a_3=0} = (k+2)(k-36ql_1^2+72ql_1+2)\times$$

$$\times[k^3(9ql_1^2-12ql_1+1)+k^2(36q^2l_1^2+36ql_1+1)+$$

$$+k(216q^2l_1^2-27ql_1^2+84ql_1-5)+3(108q^2l_1^2+6ql_1^2-36ql_1+1)]^2$$

which is the product of polynomials in k. The first two polynomials are linear with a positive coefficient of k. The third factor is the square of a third-degree polynomial. The coefficient of k^3 in this polynomial is a quadratic trinomial in l_1 and, in the domain of definition of the parameters q and l_1, it is non-negative and only vanishes for the values $q = 1/4$, $l_1 = 2/3$. This follows from the fact that the discriminant of the quadratic trinomial, which is equal to $9q(4q - 1)$, is non-positive and only vanishes when $q = 1/4$ $(m_1 = 1/2)$. So, by virtue of the fact that the coefficient of k^3 is non-negative and the coefficient if k^2 is strictly positive, we arrive at the conclusion that, with the exception of the case considered in Section 5, the curves $a_3 = 0$ and $a_6 = 0$ do not have common points when k is fairly large.

Unlike the case of secular stability, the gyroscopic stability domain is found to increase as the spring stiffness decreases. What is more, the stability domain is displaced to the left and downwards in the direction of parameter values which are characteristic of actual satellites (see Fig. 2 for $m_1 = 0.5$ and $m_1 = 0.1$; the stability domains when $k = \infty$ are shown hatched).

Both ends of the upper boundary of the stability domains are located on the upper side of the rectangular range of variation of the parameters $(m_0 = 0)$ and do not change when k changes. Actually, substitution of $m_0 = 0$ into a_6 leads to the expression

$$a_6/9\Big|_{m_0=0} = [k^2+5k+9ql_1(2-l_1)+6]^2(36ql_1^2-72ql_1+1)$$

in which the first factor is strictly positive and the second factor vanishes at the points

$$l_1 = 1 \pm \sqrt{1-\frac{1}{36q}} \quad \left(\cos^2\alpha = 1-\frac{1}{36q}\right) \tag{6.1}$$

When $q \leqslant 1/36$ (or $m_1 \leqslant m_1^* = (3-2\sqrt{2})/6 \approx 0.0286$), these points merge and disappear, and the boundary $a_6 = 0$. also disappears together with them. The stability domain is then solely determinedly the single lower left boundary $a_3 = 0$. When $m = 0{,}01$, the stability domains in Fig. 2 are positioned to the right above the curve $a_3 = 0$, which is shown for values of $k = \infty$; 5; 1.01 (the stability domain when $k = \infty$ is shown hatched).

Regardless of the value of k, the left lower boundary $a_3 = 0$ of the stability domain always starts out from the left upper corner $m_0 = 0$, $\alpha = 0$ and finishes on the right-hand side of the diagram at the point $m_0 = 12(k + 3)(k - 1)^{-1}q$. When $k = 1$, the lower left boundary of the stability domain touches the left and lower sides of the diagram and disappears. Hence, when $k \leqslant 1$, the stability domain is bounded solely by a single upper boundary. In this case, the stability domain contains values of the parameters which are characteristic of satellites (the left-hand lower corner of the diagram). In other words, if the spring stiffness is sufficiently low, the triangular stationary solutions can have applications in the dynamics of satellites.

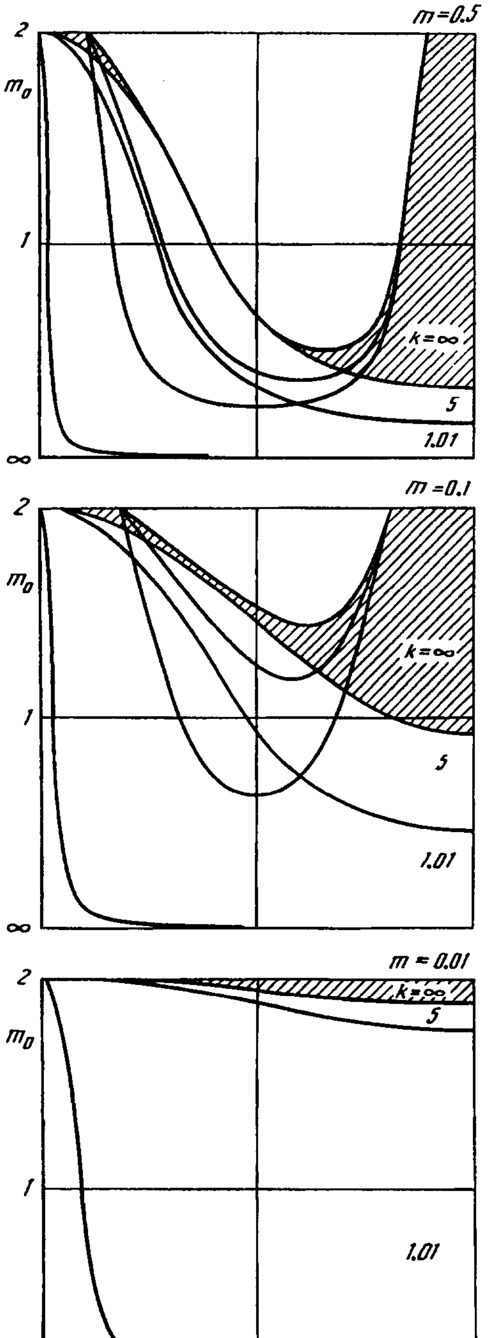

Fig. 2

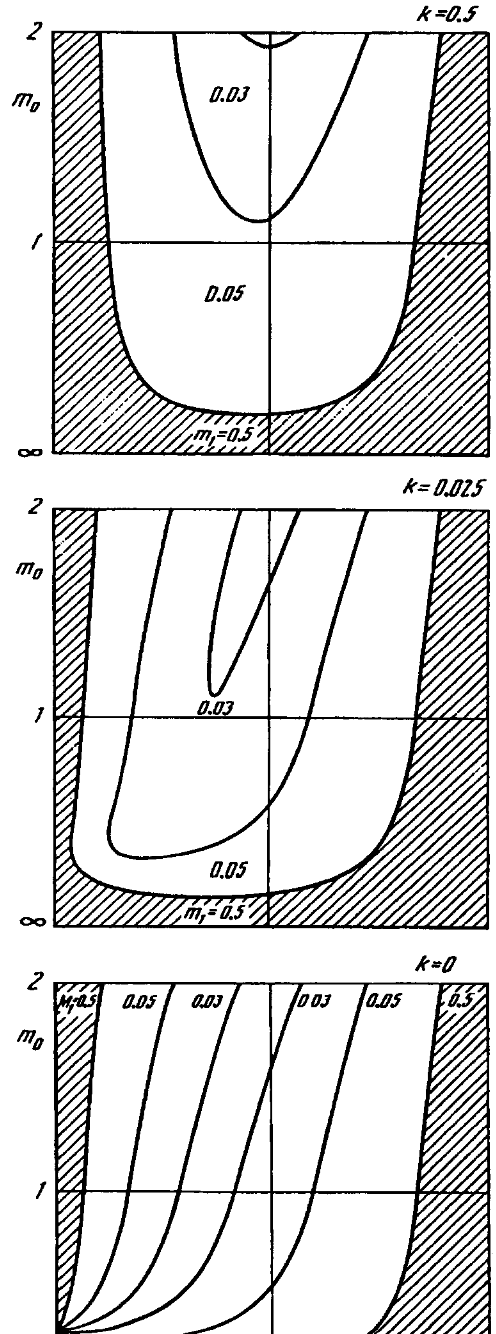

Fig. 3

In Fig. 3, it is shown, for fixed values of the coefficient of elasticity k, how the stability domains, determined by just the right-hand upper boundary $a_6 = 0$, vary as m_1 is changed (the stability domains, corresponding to $m_1 = 0.5$ are shown hatched in). In the case of the critical value $m_1 = m_1^*$, the boundary of the stability domain touches the top of the diagram and disappears. Hence, when $k \leqslant 1$ and $m_1 \leqslant m_1^*$, the stability domain fills the whole area of the stability diagram considered.

In the model considered the tension in the spring turns out to be positive and, theoretically, the joining of the masses m_1 and m_2 can be achieved using an elastic tether. However, if we take the forces of Newtonian attraction between the bodies m_1 and m_2 into consideration we observe that a positive tension only occurs when $\alpha > \pi/3$. Otherwise, additional forces, which create a tension in the tethers, are required such as, for example, two equal jet forces acting along the tether at its ends.

7. EXAMPLE: AN ESTIMATE OF THE VALUES OF THE PARAMETERS FOR A SATELLITE

We will now estimate the order of magnitude of the coefficient of elasticity corresponding to a value of $k = 1$ in the case of an Earth satellite. We consider a satellite consisting of two equal parts with a total mass $\bar{M} = \bar{m}_1 + \bar{m}_2 = 10^5$ kg ($\bar{m}_2 = \bar{m}_1$). We assume that the period of rotation in the orbit is equal to 90 min. whereupon $\bar{T} = 1/\bar{\omega} \approx 900$ s. We assume that the radius of the orbit is equal to $\bar{L} = \bar{r} = 6.4 \times 10^6$ m, the mass of the Earth is equal to $\bar{m}_0 = 6 \times 10^{24}$ kg and the distance between the masses m_1 and m_2 in the steady motion is equal to $\bar{l} = 10^5$ m.

The dimensionless values $m_1 = 1/2$, $q = 1/4$, $m_0 = 6 \times 10^{19}$, $l \approx 1.6 \times 10^{-2}$, $l_1 = l/2 \approx 0.8 \times 10^{-2}$, $e' = q/(m_0 + 1) \approx 4.2 \times 10^{-21}$, $E'' = e'k \approx 4.2 \times 10^{-21}$ correspond to these quantities.

In order to calculate the dimensional value of a parameter in the standard system of units, its dimensionless value, calculated in the special system of units of measurements chosen here, has to be multiplied by the derived unit of measurement (in the chosen special system of units), corresponding to this parameter, expressed in the standard system of units

$$\bar{E}'' = E''\bar{M}\bar{T}^{-2} \approx 0.5 \times 10^{-21} \text{ N m}^{-1}, \quad \bar{E}' = E'\bar{M}\bar{L}\bar{T}^{-2} \approx 0.5 \times 10^{-14} \text{ N}$$

For comparison, we present the value $\bar{E}_g''$ and $\bar{E}_g'$ of the derivatives of the gravitational potential $\bar{E}_g$ of the bodies m_1 and m_2 in the same steady motion $\bar{E}_g'' \approx -0.27 \times 10^{-15}$ N m^{-1} and $\bar{E}_g' \approx -0.135 \times 10^{-10}$ N. In view of the significant difference in the orders of the derivatives of the gravitational and total potentials, the coefficient of elasticity and the "tension" (negative) in the tether are of the same order of magnitude as $\bar{E}_g''$ and $\bar{E}_g'$ but of opposite sign. In particular, a tension $\bar{E}''\bar{l} \approx -0.17 \times 10^{-8}$ N is created when the length of the tether is doubled. For such small values of the coefficient of elasticity and "tension", it is advisable to speak not so much about tether but about the artificial creation of elastic forces [5].

The investigation was carried out using the REDUCE system of analytic transformations. The formulae for the kinetic and potential energy (1.1) and formulae (3.3) for the differentiation of frequently encountered expressions were used as the input data. First integrals, Routh's equations, the steady motions and the characteristic equation were obtained automatically in analytical form.

We wish to thank V. V. Beletskii for useful discussions.

This research was supported by the International Association for Promoting Cooperation with Scientists from the Independent States of the Former Soviet Union (93-1621-ext), the Russian Foundation for Basic Research (99-01-00785 and 96-15-96051) and within the framework of the Federal Special-Purpose "Integration" Programme.

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Translated by E.L.S.